AD-A054 545 WISCONSIN UNIV-MADISON MATHEMATICS RESEARCH CENTER F/G 12/1

A FUBINI THEOREM FOR ITERATED STOCHASTIC INTEGRALS.(U)

FEB 78 M A BERGER DAAG29-75-C-0024

UNCLASSIFIED MRC-TSR-1826 NL

1 OF 1

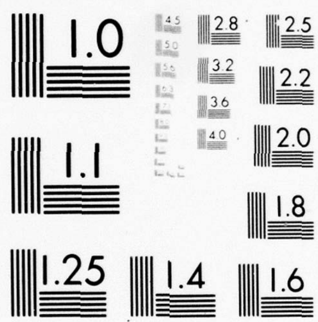
AD A054545

END

DATE FILMED

7-78

DDC



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

FOR FURTHER TRAN

12

AD A054545

MRC Technical Summary Report #1826

A FUBINI THEOREM FOR ITERATED STOCHASTIC INTEGRALS

Marc A. Berger

Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53706

February 1978

(Received December 5, 1977)

Approved for public release
Distribution unlimited

Sponsored by

U.S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

and

National Science Foundation
Washington, D. C.
20550

UNIVERSITY OF WISCONSIN - MADISON
MATHEMATICS RESEARCH CENTER

A FUBINI THEOREM FOR ITERATED STOCHASTIC INTEGRALS

Marc A. Berger

Technical Summary Report #1826
February 1978

ABSTRACT

This report extends the stochastic integral of Ito to allow for a certain class of anticipating integrands. Probabilistic and computational results concerning this extension are presented. And iterated integrals are discussed.

The motivation for this extension stems from the Ito-Volterra equation. This equation arises from feedback in the presence of white noise, and cannot be inverted using classical stochastic integrals. The inversion involving the extended integrals appears at the end of the report.

AMS(MOS) Subject Classifications: 60H20, 45D05

Key Words: Brownian Motion, Stochastic Integral, Volterra Equation.

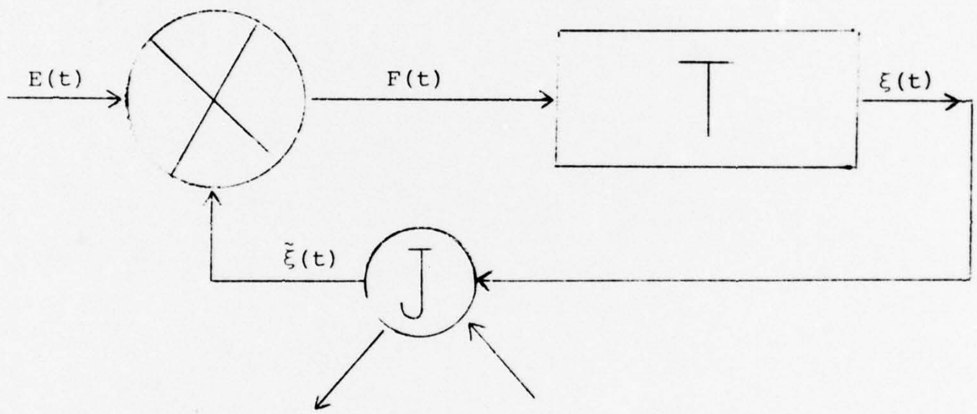
Work Unit Number 4 - Probability, Statistics and Combinatorics.

Sponsored by the United States Army under Contract No. DAAG29-75-C-0024 and by the National Science Foundation under Grant No. MCS75-17385 A01.

See 1473

SIGNIFICANCE AND EXPLANATION

In the analysis of feedback systems such as coupled electrical circuitry and economic life cycles, one is led to consider diagrams such as the one shown below. The box T signifies a transfer from the input F (e.g. current, income) to the output ξ. Junction J is a step-up or step-down point. Here either some fraction of ξ is diverted for external consumption, or else ξ is scaled up. And the remainder in the loop $\tilde{\xi}$, along with an external driving force E, is used to drive the process.



If the junction J involves a white noise (e.g. thermal noise, stock plans) then the equation governing the process involves a stochastic integral which cannot be inverted using the classical theory of stochastic integration. Thus the equation cannot be solved for ξ.

This report discusses the construction of an extension for the classical stochastic integral, designed to overcome the above limitation. Included are properties of this extension, and the subject of iterated stochastic integration.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

A FUBINI THEOREM FOR ITERATED STOCHASTIC INTEGRALS

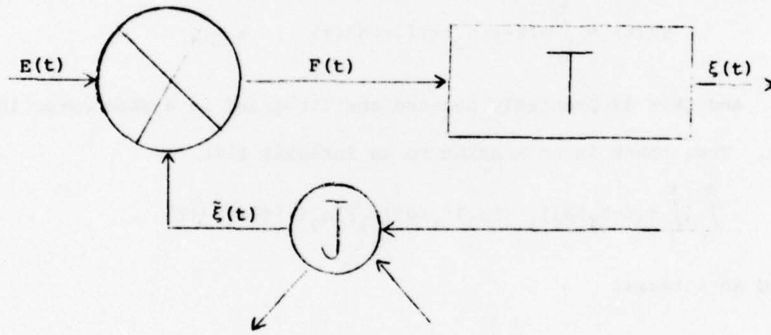
Marc A. Berger

§1. INTRODUCTION

Shown in the figure below is a typical feedback diagram. The box T signifies a transfer from the input F to the output ξ. For example,

$$\xi(t) = \int_0^t \sigma(t-\tau)F(\tau)d\tau \quad , \quad t \geq 0 \quad . \tag{1.1}$$

Junction J is a step-up or step-down point. Here either some fraction of ξ is diverted for external consumption,



or else ξ is scaled up. Thus the remainder in the loop is

$$\tilde{\xi} = \alpha\xi \quad . \tag{1.2}$$

If the process uses this remainder $\tilde{\xi}$ to drive itself, along with an external driving force E, then

$$F = E + \tilde{\xi} \quad . \tag{1.3}$$

Combining (1.1), (1.2), (1.3) it follows that the equation governing the system is

$$\xi(t) - \int_0^t \sigma(t-\tau)\alpha(\tau)\xi(\tau)d\tau = \int_0^t \sigma(t-\tau)E(\tau)d\tau, \quad t \geq 0 \quad . \tag{1.4}$$

Suppose, however, that α is in the form of a noise

Sponsored by the United States Army under Contract No. DAAG29-75-C-0024 and by the National Science Foundation under Grant No. MCS75-17385 A01.

$$\alpha = \alpha_1 + \alpha_2 z \tag{1.5}$$

where z is a white noise. Then (1.4) becomes

$$\xi(t) - \int_0^t \sigma(t-\tau)\alpha_1(\tau)\xi(\tau)d\tau - \int_0^t \sigma(t-\tau)\alpha_2(\tau)\xi(\tau)d\beta(\tau)$$

$$= \int_0^t \sigma t-\tau)E(\tau)d\tau, \quad t \geq 0 \tag{1.6}$$

where β is a Brownian motion

$$\beta(t) = \int_0^t z(\tau)d\tau \quad , \quad t \geq 0 \quad . \tag{1.7}$$

This equation is an Ito-Volterra equation, the class of which is discussed in §5. The difficulty lies in the inability to represent the iterates of the operator

$$Tf(t) = \int_0^t \sigma(t-\tau)\alpha_2(\tau)f(\tau)d\beta(\tau) \quad , \quad t \geq 0 \tag{1.8}$$

in a similar form. And this is precisely because the integrand in a stochastic integral must be nonanticipating. Thus there is no meaning to an integral like

$$\int_0^t [\int_\tau^t \sigma(t-\tau_1)\sigma(\tau_1-\tau)\alpha_2(\tau_1)d\beta(\tau_1)]\alpha_2(\tau)f(\tau)d\beta(\tau) \quad . \tag{1.9}$$

Ito [8] has defined an integral

$$I(t) = \int_0^t \int_0^t g(\tau_1,\tau_2)d\beta(\tau_1)d\beta(\tau_2)$$

where $g \in L^2([0,t] \times [0,t])$. His definition there is

$$I(t) = \int_0^t \int_0^{\tau_2} [g(\tau_1,\tau_2) + g(\tau_2,\tau_1)]d\beta(\tau_1)d\beta(\tau_2), \quad t \geq 0 \quad . \tag{1.10}$$

This integral behaves in many ways like a single stochastic integral, but <u>not</u> like two iterated integrals. For example,

$$\int_0^t \int_0^t \phi(\tau_1)\psi(\tau_2)d\beta(\tau_1)d\beta(\tau_2)$$

$$= [\int_0^t \phi(\tau)d\beta(\tau)][\int_0^t \psi(\tau)d\beta(\tau)] - \int_0^t \phi(\tau)\psi(\tau)d\tau, \quad t \geq 0 \tag{1.11}$$

for $\phi,\psi \in L^2([0,t])$. Thus, although according to (1.10) the natural definition for integrals like (1.9) should be

$$\int_0^t \int_\tau^t g(\tau,\tau_1,t)\,d\beta(\tau_1)\,d\beta(\tau)$$

(1.12)

$$= \int_0^t \int_0^\tau g(\tau_1,\tau,t)\,d\beta(\tau_1)\,d\beta(\tau) \quad , \quad t \geq 0$$

this has the disadvantage that it really is a two-dimensional integral, but <u>not</u> an iterated one-dimensional integral.

To this end we present an extension of the stochastic integral which allows one to solve equations like (1.6) by iterating operators like (1.8). This extension is, roughly speaking, the unique extension which allows integrals to be iterated one variable at a time, in the usual fashion. Thus, for example, a formula like (1.11) becomes

$$\int_0^t \int_0^t \phi(\tau_1)\psi(\tau_2)\,d\beta(\tau_1)\,d\beta(\tau_2)$$

(1.13)

$$= [\int_0^t \phi(\tau)\,d\beta(\tau)]\,[\int_0^t \psi(\tau)\,d\beta(\tau)] \quad .$$

The distinction between our integral and that of Ito is clarified through the Correction Formula (Theorem 3.A). Because of the ease of the calculations ensuing from our integral, many properties of stochastic calculus are revealed. For example, in §3 we present the Doob-Meyer decomposition for a class of nonanticipating processes. And we also present there a discussion of integrals

$$\int_0^t \beta(\lambda(\tau))\,d\beta(\tau)$$

where $\lambda(\tau) \geq \tau$. And in Theorem 4.B we provide a differentiation rule for processes

$$\xi(t) = F(t,\beta(t)) \quad , \quad t \geq 0$$

where

$$F(t,x) = \int_0^t \phi(\tau,t,x-\beta(\tau))\,d\beta(\tau) \quad , \quad t \geq 0 \; ; \; x \in R \quad .$$

For a different approach to the Correction Formula the reader is referred to Meyer [11] pp. 321-326. And for other types of random integral operators, Bharucha-Reid [4] and Tsokos and Padgett [13] are quite comprehensive.

2. ADAPTED STOCHASTIC INTEGRAL

Let $(\Omega, \mathfrak{F}, \mathbf{P})$ be a probability space, and $\{\beta(t): t \geq 0\}$ a Brownian motion on it. For $0 \leq t_1 \leq t_2$ let $\mathfrak{F}(t_1, t_2)$ denote the sub-sigma-algebra of $\mathfrak{F}$ generated by $\{\beta(\tau)-\beta(t_1): t_1 \leq \tau \leq t_2\}$. A stochastic process $\{f(t_1,t_2): 0 \leq t_1 \leq t_2\}$ is said to be L_+^2-adapted (with respect to β) if

(i) $f(\cdot, t_2)$ is separable and measurable on $[0, t_2]$, $t_2 \geq 0$

(ii) $f(t_1,t_2)$ is $\mathfrak{F}(t_1,t_2)$-measurable, $0 \leq t_1 \leq t_2$

(iii) $f(t_1,t_2) \in L^2(\Omega)$, $0 \leq t_1 \leq t_2$

(iv) $\mathbf{E}\int_0^{t_2} |f(\tau,t_2)|^2 d\tau < \infty$, $t_2 \geq 0$.

If conditions (i) and (iv) are replaced by

(i)' $f(t_1, \cdot)$ is separable and measurable on $[t_1, \infty)$, $t_1 \geq 0$

(iv)' $\mathbf{E}\int_{t_1}^{t_2} |f(t_1,\tau)|^2 d\tau < \infty$, $0 \leq t_1 \leq t_2$

then f is said to be L_-^2-adapted (with respect to β).

Ito [7] has defined the integral

$$\int_{t_1}^{t_2} f(t_1,\tau)\, d\beta(\tau)$$

for L_-^2-adapted processes f, and its properties can be found in any text on stochastic integration. (See, for example, Arnold [1] pp. 64-88, Friedman [5] pp. 59-72, Gihman and Skorohod [6] pp. 11-27, McKean [9] pp. 24-29, McShane [10] pp. 102-152, Skorohod [12] pp. 15-29.) We address ourselves to the problem of defining a new stochastic integral of the form

$$\int_{t_1}^{t_2} f(\tau,t_2)\, d\beta(\tau)$$

for L_+^2-adapted processes f.

To begin with we establish the following result characterizing L_+^2-adapted processes.

Theorem 2.A:

Let $T_n^{t_1,t_2}$ denote the region

$$\{(\tau_1,\ldots,\tau_n) : t_1 \le \tau_1 \le \cdots \le \tau_n \le t_2\} , \ 0 \le t_1 \le t_2 \ .$$

For any L_+^2-adapted process f there exists a unique sequence

$$\{\varphi_n(t_1,t_2) \in L^2(T_n^{t_1,t_2}) : n = 1,2,\ldots \ ; \ 0 \le t_1 \le t_2$$

such that, for $0 \le t_1 \le t_2$, $f(t_1,t_2)$ has the $L^2(\Omega)$ expansion

$$\mathbf{E}\, f(t_1,t_2) + \sum_{n=1}^{\infty} \int_{t_1}^{t_2}\int_{t_1}^{\tau_n} \cdots \int_{t_1}^{\tau_2} \varphi_n(t_1,t_2;\tau_1,\ldots,\tau_n)\,d\beta(\tau_1)\cdots d\beta(\tau_{n-1})\,d\beta(\tau_n) \ . \tag{2.1}$$

In particular, for $0 \le t_1 \le t_2$,

$$f(t_1,t_2) = \mathbf{E}\, f(t_1,t_2) + \int_{t_1}^{t_2} \psi(t_1,\tau,t_2)\,d\beta(\tau) \tag{2.2}$$

where $\psi(t_1,\tau,t_2)$ is $\mathfrak{F}(t_1,\tau)$-measurable, a.e. $\tau \in [t_1,t_2]$, and

$$\mathbf{E}\int_0^{\tau} |\psi(t,\tau,t_2)|^2 dt < \infty, \ t_2 \ge 0, \quad \text{a.e.} \quad \tau \in [t_1,t_2] \tag{2.3}$$

$$\mathbf{E}\int_{t_1}^{t_2} |\psi(t_1,\tau,t_2)|^2 d\tau < \infty, \ 0 \le t_1 \le t_2 \ . \tag{2.4}$$

Proof:

By considering the Brownian motion

$$\beta_*(t) = \beta(t_1+t)-\beta(t_1) \quad , \quad 0 \le t \le t_2 - t_1$$

the expansion (2.1) becomes a form of the homogeneous chaos, and follows directly from Theorem 4.2 and Theorem 5.1 of Ito [8]. The uniqueness follows from Theorem 4.3 there. The fact that f is L_+^2-adapted implies that for $t_2 \ge 0$

$$\mathbf{E}\int_0^{t_2}\int_0^{\tau}|\psi(t,\tau,t_2)|^2 dt\, d\tau$$

$$= \mathbf{E}\int_0^{t_2}\int_t^{t_2}|\psi(t,\tau,t_2)|^2 d\tau\, dt$$

$$\leq \mathbf{E}\int_0^{t_2}|f(t,t_2)|^2 dt < \infty$$

from which (2.3) follows. Similarly,

$$\mathbf{E}\int_{t_1}^{t_2}|\psi(t_1,\tau,t_2)|^2 d\tau \leq \mathbf{E}\,|f(t_1,t_2)|^2 < \infty \ .$$

■

Since, in general, nothing can be said about the existence of a formal stochastic differential $\partial_{t_1} f(t_1,t_2)$ (i.e. t_2 is held fixed), it is necessary to restrict ourselves to L_+^2-adapted processes f for which such a differential does exist. That is, we require that

(i) $\frac{\partial}{\partial t_1}\mathbf{E}\, f(t_1,t_2)$ exists, and $\frac{\partial}{\partial t_1}\varphi_n(t_1,t_2)$ exists in $L^2(T_n^{t_1,t_2})$, $0 \leq t_1 \leq t_2$; $n = 1,2,\ldots$

(ii) The series

$$\frac{\partial}{\partial t_1}\mathbf{E}\, f(t_1,t_2) + \sum_{n=1}^{\infty}\int_{t_1}^{t_2}\int_{t_1}^{\tau_n}\cdots\int_{t_1}^{\tau_2}\frac{\partial}{\partial t_1}\varphi_n(t_1,t_2;\tau_1,\ldots,\tau_n)d\beta(\tau_1)\cdots d\beta(\tau_{n-1})d\beta(\tau_n)$$

converges in $L^2(\Omega)$, for $0 \leq t_1 \leq t_2$, to an L_+^2-adapted process $f^0(t_1,t_2)$.

(iii) $\varphi_n(t_1,t_2;t_1,\cdot,\ldots,\cdot)$ exists (as a trace) in $L^2(T_{n-1}^{t_1,t_2})$, $0 \leq t_1 \leq t_2$; $n = 1,2,\ldots$

(iv) The series

$$\varphi_1(t_1,t_2;t_1) + \sum_{n=2}^{\infty}\int_{t_1}^{t_2}\int_{t_1}^{\tau_{n-1}}\cdots\int_{t_1}^{\tau_2}\varphi_n(t_1,t_2;t_1,\tau_1,\ldots,\tau_{n-1})d\beta(\tau_1)\cdots d\beta(\tau_{n-2})d\beta(\tau_{n-1})$$

converges in $L^2(\Omega)$, for $0 \leq t_1 \leq t_2$, to an L_+^2-adapted process $f^{\wedge}(t_1,t_2)$.

Such processes f are said to be <u>$L_+^{2,1}$-adapted</u> (with respect to β). The process $f\hat{}$ is called the <u>derived</u> <u>process</u> <u>of</u> <u>f</u>, and we write

$$f\hat{}(t_1,t_2) \equiv -\frac{\partial_{t_1} f(t_1,t_2)d\beta(t_1)}{dt_1} .$$

In many ways it behaves like a derivative. For example, if f is of the form

$$f(t_1,t_2) = F(\beta(t_2) - \beta(t_1)) \quad , \quad 0 \le t_1 \le t_2$$

where $F \in C^1(R)$, then

$$f\hat{}(t_1,t_2) = F'(\beta(t_2) - \beta(t_1)) \quad , \quad 0 \le t_1 \le t_2 .$$

In fact, if f is $L_+^{2,1}$-adapted, then $\partial_{t_1} f(t_1,t_2)$ formally exists and is given by

$$f^0(t_1,t_2)dt_1 - f\hat{}(t_1,t_2)d\beta(t_1) .$$

So that $f\hat{}$ is simply the negative of the diffusion part of it.

We now make the following definition. Suppose f is an $L_+^{2,1}$-adapted process of the form

$$f(t_1,t_2) = \int_{t_1}^{t_2}\int_{t_1}^{\tau_n}\cdots\int_{t_1}^{\tau_2} \varphi(t_1,t_2;\tau_1,\ldots,\tau_n)d\beta(\tau_1)\cdots d\beta(\tau_{n-1})d\beta(\tau_n)$$

where $\varphi(t_1,t_2) \in L^2(T_n^{t_1,t_2})$, $0 \le t_1 \le t_2$. Then

$$\int_{t_1}^{t_2} f(\tau,t_2)d\beta(\tau)$$

is defined to be

$$(2.5) \quad \begin{aligned} &\int_{t_1}^{t_2}\int_{t_1}^{\tau}\int_{t_1}^{\tau_n}\cdots\int_{t_1}^{\tau_2} \varphi(\tau_1,t_2;\tau_2,\ldots,\tau_n,\tau)d\beta(\tau_1)\cdots d\beta(\tau_{n-1})d\beta(\tau_n)d\beta(\tau) \\ &+ \int_{t_1}^{t_2} f\hat{}(\tau,t_2)d\tau . \end{aligned}$$

The first term here exists since f is L_+^2-adapted. We note that

$$\text{(2.6)} \qquad \mathbf{E}\int_{t_1}^{t_2} f(\tau,t_2)d\beta(\tau) = \mathbf{E}\int_{t_1}^{t_2} f^{\wedge}(\tau,t_2)d\tau$$

and that this is zero if $n > 1$. Furthermore, if $\tilde{f}$ is an $L_+^{2,1}$-adapted process of the form

$$\tilde{f}(t_1,t_2) = \int_{t_1}^{t_2}\int_{t_1}^{\tau_m}\cdots\int_{t_1}^{\tau_2} \tilde{\varphi}(t_1,t_2;\tau_1,\ldots,\tau_m)d\beta(\tau_1)\cdots d\beta(\tau_{m-1})d\beta(\tau_m)$$

where $m \leq n$ and $\tilde{\varphi}(t_1,t_2) \in L^2(T_m^{t_1,t_2})$, $0 \leq t_1 \leq t_2$, then

$$\mathbf{E}\int_{t_1}^{t_2} f(\tau,t_2)d\beta(\tau)\int_{t_1}^{t_2} \tilde{f}(\tau,t_2)d\beta(\tau)$$

$$\text{(2.7)} \quad = \begin{cases} \mathbf{E}\left[\int_{t_1}^{t_2} f(\tau,t_2)\tilde{f}(\tau,t_2)d\tau + \int_{t_1}^{t_2} f^{\wedge}(\tau,t_2)d\tau\int_{t_1}^{t_2} \tilde{f}^{\wedge}(\tau,t_2)d\tau\right], & m = n \\ \int_{t_1}^{t_2}\int_{t_1}^{\tau_{n-1}}\cdots\int_{t_1}^{\tau_1} \varphi(\tau,t_2;\tau,\tau_1,\ldots,\tau_{n-1})\tilde{\varphi}(\tau_1,t_2;\tau_2,\ldots,\tau_{n-1})d\tau d\tau_1\cdots d\tau_{n-1}, & m = n-2 \\ 0, & \text{otherwise} . \end{cases}$$

In particular,

$$\text{(2.8)} \qquad \mathbf{E}\left|\int_{t_1}^{t_2} f(\tau,t_2)d\beta(\tau)\right|^2 = \mathbf{E}\left[\int_{t_1}^{t_2} |f(\tau,t_2)|^2 d\tau + \left|\int_{t_1}^{t_2} f^{\wedge}(\tau,t_2)d\tau\right|^2\right] .$$

Using the above definition and Theorem 2.A it is now desired to extend this stochastic integral to general $L_+^{2,1}$-adapted processes f. To this end the following result is presented.

Theorem 2.B:

There is a unique extension of the integral

$$\int_{t_1}^{t_2} f(\tau,t_2)d\beta(\tau)$$

to all $L_+^{2,1}$-adapted processes f, satisfying the following continuity condition:

Whenever $\{f_n : n = 1,2,\dots\}$ is a sequence of $L_+^{2,1}$-adapted processes for which

$$\lim_{n\to\infty} \mathbf{E}\,[|f_n(t_1,t_2)|^2 + |f_n^0(t_1,t_2)|^2 + |f_n^\wedge(t_1,t_2)|^2] = 0$$

$$\int_0^{t_2} \sup_{n=1,2,\dots} \mathbf{E}\,[|f_n^0(\tau,t_2)|^2 + |f_n^\wedge(\tau,t_2)|^2]\,d\tau < \infty$$

then

$$\lim_{n\to\infty} \mathbf{E}\Big|\int_{t_1}^{t_2} f_n(\tau,t_2)\,d\beta(\tau)\Big|^2 = 0 \quad , \quad 0 \le t_1 \le t_2 \quad .$$

Proof:

For the existence of the integral it is necessary to establish the $L^2(\Omega)$ convergence of the series

(2.9) $$\sum_{n=1}^{\infty} \int_{t_1}^{t_2} f_n(\tau,t_2)\,d\beta(\tau) \quad , \quad 0 \le t_1 \le t_2 \quad ,$$

where f has the $L^2(\Omega)$ expansion

$$f(t_1,t_2) = \mathbf{E}\, f(t_1,t_2) + \sum_{n=1}^{\infty} f_n(t_1,t_2) \quad , \quad 0 \le t_1 \le t_2 \quad ,$$

and f_n is $L_+^{2,1}$-adapted and of the form

$$f_n(t_1,t_2) = \int_{t_1}^{t_2}\int_{t_1}^{\tau_n}\cdots\int_{t_1}^{\tau_2} \varphi_n(t_1,t_2;\tau_1,\dots,\tau_n)\,d\beta(\tau_1)\cdots d\beta(\tau_{n-1})\,d\beta(\tau_n),\ 0 \le t_1 \le t_2 \quad ,$$

for $n = 1,2,\dots$. Here, as before

$$\varphi_n(t_1,t_2) \in L^2(T_n^{t_1,t_2}),\ 0 \le t_1 \le t_2 \ ;\ n = 1,2,\dots \quad .$$

Consider first the series

(2.10) $$\sum_{n=1}^{\infty} \int_{t_1}^{t_2} f_n^\wedge(\tau,t_2)\,d\tau$$

By assumption, the series

$$\sum_{n=1}^{N} f_n^{\wedge}(\tau,t_2)$$

tends to $f^{\wedge}(\tau,t_2)$ in $L^2(\Omega)$ for large N, at each point $\tau \in [t_1,t_2]$. Furthermore,

$$\sum_{n=1}^{N} \mathbf{E}\,|f_n^{\wedge}(\tau,t_2)|^2 \leq \mathbf{E}\,|f^{\wedge}(\tau,t_2)|^2, \ \tau \in [t_1,t_2]$$

and, since $f^{\wedge}$ is L_+^2-adapted,

$$\int_{t_1}^{t_2} \mathbf{E}\,|f^{\wedge}(\tau,t_2)|^2 d\tau < \infty \ .$$

Thus, by the theorem on dominated convergence, the series

$$\sum_{n=1}^{N} \int_{t_1}^{t_2} \mathbf{E}\,|f_n^{\wedge}(\tau,t_2)|^2 d\tau$$

tends to $\int_{t_1}^{t_2} \mathbf{E}\,|f^{\wedge}(\tau,t_2)|^2 d\tau$ for large N. Furthermore, by the Cauchy-Schwartz inequality,

$$\mathbf{E}\,\Big|\sum_{n=1}^{N} \int_{t_1}^{t_2} f_n^{\wedge}(\tau,t_2)d\tau - \int_{t_1}^{t_2} f^{\wedge}(\tau,t_2)d\tau\Big|^2$$

$$\leq (t_2-t_1) \sum_{n=N+1}^{\infty} \int_{t_1}^{t_2} \mathbf{E}\,|f_n^{\wedge}(\tau,t_2)|^2 d\tau$$

and thus the series (2.10) converges to $\int_{t_1}^{t_2} f^{\wedge}(\tau,t_2)d\tau$ in $L^2(\Omega)$.

Consider next the series

$$\sum_{n=1}^{\infty} g_n(t_1,t_2) \tag{2.11}$$

where

$$g_n(t_1,t_2) = \int_{t_1}^{t_2}\int_{t_1}^{\tau_{n+1}} \cdots \int_{t_1}^{\tau_2} \varphi_n(\tau_1,t_2;\tau_2,\ldots,\tau_{n+1})d\beta(\tau_1)\cdots d\beta(\tau_n)d\beta(\tau_{n+1}), \ 0 \leq t_1 \leq t_2 \ .$$

Since f_n^0 is L_+^2-adapted it follows that

$$g_n(t_1,t_2) = \int_{t_1}^{t_2}\int_{t_1}^{\tau_n}\cdots\int_{t_1}^{\tau_2} [\beta(\tau_1)-\beta(t_1)]\varphi_n(t_1,t_2;\tau_1,\dots,\tau_n)d\beta(\tau_1)\cdots d\beta(\tau_{n-1})d\beta(\tau_n)$$

$$+ \int_{t_1}^{t_2}\int_{\tau}^{t_2}\int_{\tau}^{\tau_n}\cdots\int_{\tau}^{\tau_2} [\beta(\tau_1)-\beta(\tau)]\frac{\partial}{\partial\tau}\varphi_n(\tau,t_2;\tau_1,\dots,\tau_n)d\beta(\tau_1)\cdots d\beta(\tau_{n-1})d\beta(\tau_n)d\tau$$

and thus by the Cauchy-Schwartz inequality

$$\mathbf{E}\,|g_n(t_1,t_2)|^2 \le 2(t_2-t_1)\mathbf{E}\,[|f_n(t_1,t_2)|^2 + (t_2-t_1)\int_{t_1}^{t_2} |f_n^0(\tau,t_2)|^2 d\tau] \ .$$

Since f^0 is L_+^2-adapted, the theorem on dominated convergence can be used as before to show that

$$\sum_{n=1}^{N}\int_{t_1}^{t_2} \mathbf{E}\,|f_n^0(\tau,t_2)|^2 d\tau$$

tends to $\int_{t_1}^{t_2} \mathbf{E}\,|f^0(\tau,t_2)|^2 d\tau$ for large N. Hence the series (2.11) converges in $L^2(\Omega)$. And now using (2.5) it follows that the series (2.9) also converges.

Concerning the continuity condition, the estimate

$$\mathbf{E}\,\left|\int_{t_1}^{t_2} f(\tau,t_2)d\beta(\tau)\right|^2 \le 2\ \mathbf{E}\,\left|\int_{t_1}^{t_2} f^{\wedge}(\tau,t_2)d\tau\right|^2$$

$$+ 4(t_2-t_1)\mathbf{E}\,|f(t_1,t_2)|^2$$

$$+ 4(t_2-t_1)^2\mathbf{E}\,\left|\int_{t_1}^{t_2} f^0(\tau,t_2)d\tau\right|^2$$

shows that the integral we constructed satisfies this condition.

∎

Some of the important properties of this integral are summarized in the following result.

Theorem 2.C:

Let f_a, f_b be $L_+^{2,1}$-adapted, and set

$$I_a(t_1,t_2) = \int_{t_1}^{t_2} f_a(\tau,t_2)d\beta(\tau), \quad 0 \le t_1 \le t_2$$

$$I_b(t_1,t_2) = \int_{t_1}^{t_2} f_b(\tau,t_2)d\beta(\tau), \quad 0 \le t_1 \le t_2 \ .$$

Let $a,b \in R$. Then

(a) (Linearity) $\int_{t_1}^{t_2} [af_a(\tau,t_2)+bf_b(\tau,t_2)]d\beta(\tau) = aI_a(t_1,t_2) + bI_b(t_1,t_2)$

(b) (Smoothness) $I_a(t_1,t_2)$ is $L_+^{2,1}$-adapted, and $I_a^0 = 0$, $I_a^{\hat{}} = f$

(c) $\mathbf{E}\, I_a(t_1,t_2) = \mathbf{E} \int_{t_1}^{t_2} f^{\hat{}}(\tau,t_2)d\tau$

(d) $\mathbf{E}\, I_a(t_1,t_2)I_b(t_1,t_2)$

$$= \mathbf{E}\,[\int_{t_1}^{t_2} f_a(\tau,t_2)f_b(\tau,t_2)d\tau + \int_{t_1}^{t_2} f_a^{\hat{}}(\tau,t_2)d\tau \int_{t_1}^{t_2} f_b^{\hat{}}(\tau,t_2)d\tau]$$

$$+ \mathbf{E} \int_{t_1}^{t_2} [f_a^{\hat{}}(\tau,t_2)g_b(\tau,t_2) + f_b^{\hat{}}(\tau,t_2)g_a(\tau,t_2)]d\tau$$

$$= \mathbf{E} \int_{t_1}^{t_2} f_a(\tau,t_2)f_b(\tau,t_2)d\tau$$

$$+ \mathbf{E} \int_{t_1}^{t_2} [f_a^{\hat{}}(\tau,t_2)I_b(\tau,t_2) + f_b^{\hat{}}(\tau,t_2)I_a(\tau,t_2)]d\tau$$

where

$$g_a(\tau,t_2) = \int_{\tau}^{t_2} \mathbf{E}\, f_a(\tau_1,t_2)d\beta(\tau_1)$$

$$+ \sum_{n=1}^{\infty} \int_{\tau}^{t_2} \int_{\tau}^{\tau_{n+1}} \cdots \int_{\tau}^{\tau_2} \varphi_{a,n}(\tau_1,t_2;\tau_2,\dots,\tau_{n+1})d\beta(\tau_1)\cdots d\beta(\tau_n)d\beta(\tau_{n+1})$$

and the functions $\{\varphi_{a,n} : n = 1,2,\ldots\}$ are as in Theorem 2.A. And g_b is defined analogously.

In particular,

$$\mathbf{E}\,|I_a(t_1,t_2)|^2 = \mathbf{E}\,[\int_{t_1}^{t_2} |f_a(\tau,t_2)|^2 d\tau + |\int_{t_1}^{t_2} \hat{f}_a(\tau,t_2)d\tau|^2]$$

$$+ 2\mathbf{E}\int_{t_1}^{t_2} \hat{f}_a(\tau,t_2)g_a(\tau,t_2)d\tau$$

$$= \mathbf{E}\int_{t_1}^{t_2} |f_a(\tau,t_2)|^2 d\tau + 2\mathbf{E}\int_{t_1}^{t_2} \hat{f}_a(\tau,t_2)I_a(\tau,t_2)d\tau \quad .$$

Proof:

All four parts follow directly from Theorem 2.A, using (2.5), (2.6), (2.7), (2.8) and the observation

$$g_a(t_1,t_2) = I_a(t_1,t_2) - \int_{t_1}^{t_2} \hat{f}_a(\tau,t_2)d\tau$$

and similarly for g_b.

■

§3. CORRECTION FORMULA

In this section we present the following

Theorem 3.A (Correction Formula):

Let f be $L_+^{2,1}$-adapted, and assume

$$\int_{t_1}^{t_2} |f(t,t)|^2 dt < \infty$$

$$\mathbf{E}\int_{t_1}^{t_2}\int_{t_1}^{t} |f(\tau,t)|^2 d\tau \, dt < \infty \ .$$

Then

$$\int_{t_1}^{t_2}\int_{\tau}^{t_2} f(\tau,t)d\beta(t)d\beta(\tau)$$

$$= \int_{t_1}^{t_2}\int_{t_1}^{t} f(\tau,t)d\beta(\tau)d\beta(t) + \int_{t_1}^{t_2} f(t,t)dt \ .$$

Proof:

If f is a deterministic function, the result follows directly from (2.5). So let f be of the form

$$(3.1) \quad f(\tau,t) = \int_{\tau}^{t}\int_{\tau}^{\tau_n}\cdots\int_{\tau}^{\tau_2} \varphi(\tau,t;\tau_1,\ldots,\tau_n)d\beta(\tau_1)\cdots d\beta(\tau_{n-1})d\beta(\tau_n), \quad t_1 \le \tau \le t \le t_2 \ ,$$

where $n \geq 1$. Then by (2.5)

$$\int_{t_1}^{t} f(\tau,t)d\beta(\tau) = \int_{t_1}^{t}\int_{t_1}^{\tau_{n+1}}\cdots\int_{t_1}^{\tau_2} \varphi(\tau_1,t;\tau_2,\ldots,\tau_{n+1})d\beta(\tau_1)\cdots d\beta(\tau_n)d\beta(\tau_{n+1})$$

$$+ \int_{t_1}^{t} f^{\wedge}(\tau,t)d\tau \qquad , \qquad t_1 \le t \le t_2 \ ,$$

and thus

$$\int_{t_1}^{t_2}\int_{t_1}^{t} f(\tau,t)d\beta(\tau)d\beta(t)$$

$$= \int_{t_1}^{t_2}\int_{t_1}^{\tau}\int_{t_1}^{\tau_{n+1}}\cdots\int_{t_1}^{\tau_1} \varphi(\tau_1,\tau;\tau_2,\ldots,\tau_{n+1})d\beta(\tau_1)\cdots d\beta(\tau_n)d\beta(\tau_{n+1})d\beta(\tau)$$

$$+ \int_{t_1}^{t_2}\int_{\tau}^{t_2} f^{\hat{}}(\tau,t)d\beta(t)d\tau \quad .$$

Next let

$$g(\tau,t_2) = \int_{\tau}^{t_2} f(\tau,t)d\beta(t) \quad , \quad t_1 \le \tau \le t_2 \quad ,$$

so that

$$g(\tau,t_2) = \int_{\tau}^{t_2}\int_{\tau}^{\tau_{n+1}}\cdots\int_{\tau}^{\tau_2} \varphi(\tau,\tau_{n+1};\tau_1,\ldots,\tau_n)d\beta(\tau_1)\cdots d\beta(\tau_n)d\beta(\tau_{n+1}), \ t_1 \le \tau \le t_2 \quad .$$

Then by (2.5)

$$\int_{t_1}^{t_2}\int_{\tau}^{t_2} f(\tau,t)d\beta(t)d\beta(\tau)$$

$$= \int_{t_1}^{t_2}\int_{t_1}^{\tau}\int_{t_1}^{\tau_{n+1}}\cdots\int_{t_1}^{\tau_2} \varphi(\tau_1,\tau;\tau_2,\ldots,\tau_{n+1})d\beta(\tau_1)\cdots d\beta(\tau_n)d\beta(\tau_{n+1})d\beta(\tau)$$

$$+ \int_{t_1}^{t_2} g^{\hat{}}(\tau,t_2)d\tau \quad .$$

Since $f(t,t) \equiv 0$ it is enough to show that

$$g^{\hat{}}(\tau,t_2) = \int_{\tau}^{t_2} f^{\hat{}}(\tau,t)d\beta(t), \ t_1 \le \tau \le t_2 \quad ,$$

and this is clear by inspection. Thus the Correction Formula holds if f is of the form (3.1). Finally, using Theorem 2.A and the continuity condition of Theorem 2.B, it follows that the Correction Formula holds for any $L_+^{2,1}$-adapted process f.

∎

The process

$$\eta(t_1,t_2) = \int_{t_1}^{t_2} f(\tau,t_2)d\beta(\tau)$$

is, by Theorem 2.C, $L_+^{2,1}$-adapted, and, as such, possesses a formal differential

$$\partial_{t_1}\eta(t_1,t_2) \quad .$$

However, it is of greater interest to compute

$$\partial_{t_2}\eta(t_1,t_2)$$

since this is an Ito-differential (not just a formal notation), and

$$\eta(t_1,t_2) = \int_{t_1}^{t_2} \partial_\tau \eta(t_1,\tau) \quad .$$

The Correction Formula can be employed to this end.

Theorem 3.B:

Let f be $L_+^{2,1}$-adapted and set

$$\eta(t_1,t_2) = \int_{t_1}^{t_2} f(\tau,t_2)d\beta(\tau) \quad .$$

Suppose

$$\begin{aligned} &f(t_1,t_2) - f(t_1,t_1) \\ &= \int_{t_1}^{t_2} a(t_1,\tau)d\tau + \int_{t_1}^{t_2} b(t_1,\tau)d\beta(\tau), \; 0 \le t_1 \le t_2 \end{aligned}$$

where a, b are $L_+^{2,1}$-adapted processes satisfying

$$\mathbf{E}\int_0^T \int_0^{t_2} [|a(t_1,t_2)|^2 + |b(t_1,t_2)|^2]dt_1 dt_2 < \infty \quad , \quad T \ge 0$$

$$\int_0^T |b(t,t)|^2 dt < \infty \qquad , \qquad T \ge 0 \quad .$$

Then

$$\partial_{t_2}\eta(t_1,t_2) = [b(t_2,t_2) + \int_{t_1}^{t_2} a(\tau,t_2)d\beta(\tau)]dt_2$$

$$+ [f(t_2,t_2) + \int_{t_1}^{t_2} b(\tau,t_2)d\beta(\tau)]d\beta(t_2) \quad , \quad 0 \le t_1 \le t_2 \quad .$$

Proof:

The theorem follows directly from the Correction Formula. Indeed,

$$\int_{t_1}^{t_2} [b(\tau,\tau) + \int_{t_1}^{\tau} a(\tau',\tau)d\beta(\tau')]d\tau$$

$$+ \int_{t_1}^{t_2} [f(\tau,\tau) + \int_{t_1}^{\tau} b(\tau',\tau)d\beta(\tau')]d\beta(\tau)$$

$$= \int_{t_1}^{t_2} f(\tau',\tau')d\beta(\tau')$$

$$+ \int_{t_1}^{t_2} [\int_{\tau'}^{t_2} a(\tau',\tau)d\tau + \int_{\tau'}^{t_2} b(\tau',\tau)d\beta(\tau)]d\beta(\tau')$$

$$= \int_{t_1}^{t_2} f(\tau',t_2)d\beta(\tau') \quad .$$

■

It is worthy of note that although the process (here t_1 is fixed)

$$x(t) = \int_{t_1}^{t} f(\tau,t)d\beta(\tau)$$

is not a martingale, the Correction Formula does provide its Doob-Meyer decomposition. Thus if

$$f(t_1,t_2) = \int_{t_1}^{t_2} \psi(t_1,\tau,t_2)d\beta(\tau)$$

(cf. Theorem 2.A), then

$$\int_{t_1}^{t} f(\tau,t)d\beta(\tau)$$

(3.2)
$$= \int_{t_1}^{t} [\int_{t_1}^{\tau} \psi(\tau_1,\tau,t)d\beta(\tau_1)]d\beta(\tau)$$

$$+ \int_{t_1}^{t} \psi(\tau,\tau,t)d\tau \quad .$$

For example, the decomposition for

$$\beta(t) \int_{0}^{t} f(\tau)d\beta(\tau)$$

is

$$\int_{0}^{t} [\beta(\tau)f(\tau) + \int_{0}^{\tau} f(\tau_1)d\beta(\tau_1)]d\beta(\tau) + \int_{0}^{t} f(\tau)d\tau \quad .$$

As another application of the Correction Formula, let $\lambda(t)$ be a strictly increasing differentiable function of t on $[t_1,t_2]$ with

(3.3)
$$\lambda(t) > t \quad , \quad t_1 \leq t \leq t_2 \quad .$$

Suppose we were to define, for $t_1 \leq \tau \leq t \leq t_2$,

$$f(\tau,t) = \begin{cases} 1 & , \quad t \leq \lambda(\tau) \\ 0 & , \quad t > \lambda(\tau) \end{cases}$$

and substitute this in the Correction Formula. Then

(3.4)
$$\int_{t_1}^{\lambda^{-1}(t_2)} \beta(\lambda(\tau))d\beta(\tau) + \int_{\lambda(t_1)}^{t_2} \beta(\lambda^{-1}(\tau))d\beta(\tau)$$

$$= \beta(t_2)\beta(\lambda^{-1}(t_2)) - \beta(t_1)\beta(\lambda(t_1)) \quad .$$

This is an integration by parts formula. Of course the difficulty here is that f is not $L_+^{2,1}$-adapted. But in this case (since f is deterministic) the Correction Formula can be verified directly from (2.5). In fact, as long as the process

$$g(\tau,t_2) = \int_{\tau}^{t_2} f(\tau,t)d\beta(\tau) \quad , \quad t_1 \le \tau \le t_2 \ ,$$

has a derived process $g^\wedge$, then

$$\int_{t_1}^{t_2} g(\tau,t_2)d\beta(\tau)$$

$$= \int_{t_1}^{t_2}\int_{t_1}^{t} f(\tau,t)d\beta(\tau)d\beta(t) + \int_{t_1}^{t_2} g^\wedge(\tau,t_2)d\tau \ .$$

Now we check that

$$g(\tau,t_2) = \begin{cases} \beta(\lambda(\tau)) - \beta(\tau) & , \quad t_1 \le \tau \le \lambda^{-1}(t_2) \\ \beta(t_2) - \beta(\tau) & , \quad \lambda^{-1}(t_2) \le \tau \le t_2 \ . \end{cases}$$

Because of (3.3) it follows that $g^\wedge \equiv 1$. Thus (3.4) is established. However, a more difficult question involves the case where

$$\lambda(t) \ge t \quad , \quad t_1 \le t \le t_2 \ , \tag{3.5}$$

and the strict inequality (3.3) no longer holds. Here we have

$$g^\wedge(\tau,t_2) = \begin{cases} 1 & , \quad \lambda(\tau) \ne \tau \\ 1 - (1 \wedge \lambda'(\tau)) & , \quad \lambda(\tau) = \tau \ . \end{cases}$$

Thus we arrive at the following extension of (3.3).

$$\int_{t_1}^{\lambda^{-1}(t_2)} \beta(\lambda(\tau))d\beta(\tau) + \int_{\lambda(t_1)}^{t_2} \beta(\lambda^{-1}(\tau))d\beta(\tau)$$

$$= \beta(t_2)\beta(\lambda^{-1}(t_2)) - \beta(t_1)\beta(\lambda(t_1)) \tag{3.6}$$

$$- \int_A (1 \wedge \lambda'(\tau))d\tau$$

where A is the set $\{\tau \in [t_1,t_2]:\lambda(\tau) = \tau\}$. Now we merely note that

$$\int_A (1 \wedge \lambda'(\tau))d\tau = \int_A d\tau$$

and we arrive at the following:

Theorem 3.C (Integration by Parts):

Let $\lambda(t)$ be a strictly increasing differentiable function of t on $[t_1,t_2]$, with

$$\lambda(t) \geq t \quad , \quad t_1 \leq t \leq t_2 \ .$$

Let A be the set $\{t \in [t_1,t_2]:\lambda(t) = t\}$. Then

$$\int_{t_1}^{\lambda^{-1}(t_2)} \beta(\lambda(\tau))d\beta(\tau) + \int_{\lambda(t_1)}^{t_2} \beta(\lambda^{-1}(\tau))d\beta(\tau)$$

$$= \beta(t_2)\beta(\lambda^{-1}(t_2)) - \beta(t_1)\beta(\lambda(t_1)) - \mathcal{L}(A)$$

where $\mathcal{L}$ is Lebesgue measure.

Similar techniques like those used to establish Theorem 3.C can be used to generalize the Correction Formula for functions f defined on

$$S = \{(\tau,t):\lambda_1(\tau) \wedge t_2 \leq t \leq \lambda_2(\tau) \wedge t_2\}$$

where λ_1, λ_2 satisfy the conditions of Theorem 3.C, and $\lambda_1 \leq \lambda_2$. We merely extend the function f defined on S to the whole triangle, $t_1 \leq \tau \leq t \leq t_2$, by setting it to zero on the complement of S. The reader can check that

$$\iint_S f(\tau,t)d\beta(t)d\beta(\tau)$$

$$= \iint_S f(\tau,t)d\beta(\tau)d\beta(t) + \int_A f(t,t)dt \tag{3.7}$$

where

$$A = S \cap \{(\tau,t):\tau = t\} \ .$$

§4. CARATHEODORY PRINCIPLE

A particularly interesting class of stochastic processes are those of the form

$$f(t_1,t_2) = \phi(t_1,t_2,\beta(t_2) - \beta(t_1)) \quad , \quad 0 \le t_1 \le t_2 \quad .$$

The conditions for f to be L_+^2-adapted are

$$\int_{-\infty}^{\infty} |\phi(t_1,t_2,x)|^2 e^{-\frac{x^2}{2(t_2-t_1)}} dx < \infty \quad , \quad 0 \le t_1 \le t_2 \tag{4.1}$$

$$\int_{-\infty}^{\infty}\int_0^{t_2} \frac{|\phi(\tau,t_2,x)|^2}{\sqrt{t_2-\tau}} e^{-\frac{x^2}{2(t_2-\tau)}} d\tau\, dx < \infty \quad , \quad t_2 \ge 0 \quad . \tag{4.2}$$

And the conditions for f to be $L_+^{2,1}$-adapted are that the functions

$$\frac{\partial}{\partial x} \phi(t_1,t_2,x) \quad , \quad \left(\frac{\partial}{\partial t_1} + \frac{1}{2}\frac{\partial^2}{\partial x^2}\right) \phi(t_1,t_2,x)$$

also satisfy (4.1) and (4.2). For such processes f the integral

$$\int_{t_1}^{t_2} f(\tau,t_2)\, d\beta(\tau)$$

can be related to an Ito stochastic integral. In fact we have the following result:

Theorem 4.A (Caratheodory Principle):

Let f be an $L_+^{2,1}$-adapted process of the form

$$f(t_1,t_2) = \phi(t_1,t_2,\beta(t_2) - \beta(t_1)) \quad , \quad 0 \le t_1 \le t_2 \quad .$$

Then

$$\int_{t_1}^{t_2} f(\tau,t_2)\, d\beta(\tau) = F(t_1,t_2,\beta(t_2)) \quad , \quad 0 \le t_1 \le t_2 \quad ,$$

where

$$F(t_1,t_2,x) = \int_{t_1}^{t_2} \phi(\tau,t_2,x - \beta(\tau))\, d\beta(\tau) \quad , \quad 0 \le t_1 \le t_2 \; ; \; x \in R \quad .$$

The proof relies on the following two lemmas.

Lemma I:

Let H_n be the Hermite polynomial* of degree n, where $n \geq 1$. And let $a(t_1,t_2)$ be differentiable in t_1 and satisfy

$$\int_0^{t_2} [|a(\tau,t_2)|^2 + |\frac{\partial}{\partial\tau} a(\tau,t_2)|^2] d\tau < \infty \ , \ t_2 \geq 0 \quad .$$

Then

$$\int_{t_1}^{t_2} a(\tau,t_2) H_n(t_2-\tau, \beta(t_2)-\beta(\tau)) d\beta(\tau)$$

$$= \frac{1}{n+1} a(t_1,t_2) H_{n+1}(t_2-t_1, \beta(t_2)-\beta(t_1))$$

$$+ \frac{1}{n+1} \int_{t_1}^{t_2} \frac{\partial}{\partial\tau} a(\tau,t_2) H_{n+1}(t_2-\tau, \beta(t_2)-\beta(\tau)) d\tau$$

$$+ n \int_{t_1}^{t_2} a(\tau,t_2) H_{n-1}(t_2-\tau, \beta(t_2)-\beta(\tau)) d\tau \ , \ 0 \leq t_1 \leq t_2 \quad .$$

Lemma II:

Theorem 4.A holds for functions ϕ of the form

$$\phi(t_1,t_2,x) = a(t_1,t_2) H_n(t_2-t_1,x)$$

where a satisfies the conditions of Lemma I.

Proof of Lemma I:

The proof relies on the fact that

$$\frac{1}{n!} H_n(t_2-t_1, \beta(t_2)-\beta(t_1))$$

(4.3)
$$= \int_{t_1}^{t_2} \int_{t_1}^{\tau_n} \cdots \int_{t_1}^{\tau_2} d\beta(\tau_1) \cdots d\beta(\tau_{n-1}) d\beta(\tau_n) \ , \ n = 1,2,\ldots \quad .$$

---

*These polynomials are defined by

$$H_n(t,x) = (-t)^n e^{\frac{x^2}{2t}} \frac{\partial^n}{\partial x^n} e^{-\frac{x^2}{2t}} \ , \ n = 0,1,\ldots \ ; \ t \geq 0; \ x \in R \quad .$$

(A very short proof of this result appears in McKean [9] p. 37.) Thus, by (2.5)

$$\frac{1}{n!}\int_{t_1}^{t_2} a(\tau,t_2)H_n(t_2-\tau,\beta(t_2)-\beta(\tau))d\beta(\tau)$$

$$= \int_{t_1}^{t_2}\int_{t_1}^{\tau_{n+1}}\cdots\int_{t_1}^{\tau_2} a(\tau_1,t_2)d\beta(\tau_1)\cdots d\beta(\tau_n)d\beta(\tau_{n+1})$$

$$+ \int_{t_1}^{t_2} a(\tau,t_2)\int_{\tau}^{t_2}\int_{\tau}^{\tau_{n-1}}\cdots\int_{\tau}^{\tau_2} d\beta(\tau_1)\cdots d\beta(\tau_{n-2})d\beta(\tau_{n-1})d\tau$$

$$= a(t_1,t_2)\int_{t_1}^{t_2}\int_{t_1}^{\tau_{n+1}}\cdots\int_{t_1}^{\tau_2} d\beta(\tau_1)\cdots d\beta(\tau_n)d\beta(\tau_{n+1})$$

$$+ \int_{t_1}^{t_2}\frac{\partial}{\partial\tau}a(\tau,t_2)\int_{\tau}^{t_2}\int_{\tau}^{\tau_{n+1}}\cdots\int_{\tau}^{\tau_2} d\beta(\tau_1)\cdots d\beta(\tau_n)d\beta(\tau_{n+1})d\tau$$

$$+ \int_{t_1}^{t_2} a(\tau,t_2)\int_{\tau}^{t_2}\int_{\tau}^{\tau_{n-1}}\cdots\int_{\tau}^{\tau_2} d\beta(\tau_1)\cdots d\beta(\tau_{n-2})d\beta(\tau_{n-1})d\tau$$

$$= \frac{1}{(n+1)!}a(t_1,t_2)H_{n+1}(t_2-t_1,\beta(t_2)-\beta(t_1))$$

$$+ \frac{1}{(n+1)!}\int_{t_1}^{t_2}\frac{\partial}{\partial\tau}a(\tau,t_2)H_{n+1}(t_2-\tau,\beta(t_2)-\beta(\tau))d\tau$$

$$+ \frac{1}{(n-1)!}\int_{t_1}^{t_2} a(\tau,t_2)H_{n-1}(t_2-\tau,\beta(t_2)-\beta(\tau))d\tau$$

from which the desired result follows.

■

Proof of Lemma II:

Let

$$\phi_\lambda(t_1,t_2,x) = a(t_1,t_2)e^{\lambda x - \frac{1}{2}\lambda^2(t_2-t_1)},\quad 0 \le t_1 \le t_2\ ;\ \lambda,x \in R\ .$$

Then (evaluate the integral!)

$$F_\lambda(t_1,t_2,x) = \int_{t_1}^{t_2} \phi_\lambda(\tau,t_2,x-\beta(\tau))\,d\beta(\tau)$$

$$= \frac{1}{\lambda}\phi_\lambda(t_1,t_2,x-\beta(t_1)) - \frac{1}{\lambda}a(t_2,t_2)e^{\lambda[x-\beta(t_2)]}$$

$$+ \lambda\int_{t_1}^{t_2} \phi_\lambda(t_2-\tau,x-\beta(\tau))\,d\tau$$

$$+ \frac{1}{\lambda}\int_{t_1}^{t_2} \frac{\partial}{\partial\tau}a(\tau,t_2)e^{\lambda[x-\beta(\tau)] - \frac{1}{2}\lambda^2(t_2-\tau)}\,d\tau \ , \ 0 \le t_1 \le t_2 \ ; \ x \in R$$

and thus

$$F_\lambda(t_1,t_2,\beta(t_2)) = \frac{1}{\lambda}\phi_\lambda(t_1,t_2,\beta(t_2)-\beta(t_1)) - \frac{1}{\lambda}a(t_1,t_2)$$

$$+ \lambda\int_{t_1}^{t_2} \phi_\lambda(t_2-\tau,\beta(t_2)-\beta(\tau))\,d\tau$$

$$+ \frac{1}{\lambda}\int_{t_1}^{t_2} \frac{\partial}{\partial\tau}a(\tau,t_2)\{e^{\lambda[\beta(t_2)-\beta(\tau)] - \frac{1}{2}\lambda^2(t_2-\tau)}-1\}\,d\tau, \quad 0 \le t_1 \le t_2 \ .$$

On the other hand,

$$\phi_\lambda(t_1,t_2,x) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} a(t_1,t_2)H_n(t_2-t_1,x) \ , \ 0 \le t_1 \le t_2 \ ; \ x \in R$$

and thus, by Lemma I,

$$\int_{t_1}^{t_2} \phi_\lambda(\tau,t_2,\beta(t_2)-\beta(\tau))\,d\beta(\tau)$$

$$= \frac{1}{\lambda}\phi_\lambda(t_1,t_2,\beta(t_2)-\beta(t_1)) - \frac{1}{\lambda}a(t_1,t_2)$$

$$+ \frac{1}{\lambda}\int_{t_1}^{t_2} \frac{\partial}{\partial\tau}a(\tau,t_2)\{e^{\lambda[\beta(t_2)-\beta(\tau)] - \frac{1}{2}\lambda^2(t_2-\tau)}-1\}\,d\tau$$

$$+ \lambda\int_{t_1}^{t_2} \phi_\lambda(t_2-\tau,\beta(t_2)-\beta(\tau))\,d\tau \ , \ 0 \le t_1 \le t_2 \ .$$

And from this it follows that Theorem 4.A holds for $\{\phi_\lambda : \lambda \in R\}$; that is,

$$\int_{t_1}^{t_2} \phi_\lambda(t_2-\tau,\beta(t_2)-\beta(\tau))d\beta(\tau) = F_\lambda(t_1,t_2,\beta(t_2)) \quad , \quad 0 \le t_1 \le t_2 \ ; \ \lambda \in R \ .$$

By differentiating this equation n times with respect to λ, and setting $\lambda = 0$, it follows that Theorem 4.A holds for the function

$$\phi(t_1,t_2,x) = a(t_1,t_2)H_n(t_2-t_1,x) \quad , \quad 0 \le t_1 \le t_2 \ ; \ x \in R \ .$$

■

Now we are in a position to present the

Proof of Theorem 4.A:

Let ϕ satisfy (4.1). Then, because of the completeness of the Hermite polynomials, there exists a unique sequence $\{a_n(t_1,t_2) : n = 0,1,\dots \ ; \ 0 \le t_1 \le t_2$ such that

$$\phi(t_1,t_2,x) = \sum_{n=0}^{\infty} a_n(t_1,t_2)H_n(t_2-t_1,x) \quad , \quad 0 \le t_1 \le t_2 \ ; \ x \in R$$

in the sense that

$$\lim_{N\to\infty} \int_{-\infty}^{\infty} \left|\phi(t_1,t_2,x) - \sum_{n=0}^{N} a_n(t_1,t_2)H_n(t_2-t_1,x)\right|^2 e^{-\frac{x^2}{2(t_2-t_1)}} dx = 0 \quad , \quad 0 \le t_1 \le t_2 \ .$$

Furthermore, since

$$\frac{\partial}{\partial x} H_n(t,x) = n \, H_{n-1}(t,x) \quad , \quad n = 1,2,\dots \ ; \ t \ge 0; \ x \in R$$

it follows that if $\frac{\partial}{\partial x}\phi(t_1,t_2,x)$ satisfies (4.1)

then

$$\frac{\partial}{\partial x}\phi(t_1,t_2,x) = \sum_{n=0}^{\infty} a_n(t_1,t_2)\frac{\partial}{\partial x} H_n(t_2-t_1,x) \quad , \quad 0 \le t_1 \le t_2 \ ; \ x \in R$$

in the same sense. Finally, since

$$\left(\frac{\partial}{\partial t} - \frac{1}{2}\frac{\partial^2}{\partial x^2}\right)H_n(t,x) = - \, n(n-1)H_{n-2}(t,x) \quad , \quad t \ge 0; \ x \in R$$

it likewise follows that if $\left(\frac{\partial}{\partial t_1} + \frac{1}{2}\frac{\partial^2}{\partial x^2}\right)\phi(t_1,t_2,x)$ satisfies (4.1) then

$$\left(\frac{\partial}{\partial t_1} + \frac{1}{2}\frac{\partial^2}{\partial x^2}\right)\phi(t_1,t_2,x) = \sum_{n=0}^{\infty}\left(\frac{\partial}{\partial t_1} + \frac{1}{2}\frac{\partial^2}{\partial x^2}\right)a_n(t_1,t_2)H_n(t_2-t_1,x) ,$$

$$0 \le t_1 \le t_2 ; x \in R$$

in the same sense.

Now we define for $n = 0,1,\ldots$

$$f_n(t_1,t_2) = a_n(t_1,t_2)H_n(t_2-t_1,\beta(t_2)-\beta(t_1)) , 0 \le t_1 \le t_2 .$$

Because of (4.2) and the continuity condition of Theorem 2.B it follows that, for $0 \le t_1 \le t_2$,

$$\sum_{n=0}^{N}\int_{t_1}^{t_2} f_n(\tau,t_2)d\beta(\tau)$$

tends to $\int_{t_1}^{t_2} f(\tau,t_2)d\beta(\tau)$ in $L^2(\Omega)$ for large N. Furthermore, if

$$F_n(t_1,t_2,x) = \int_{t_1}^{t_2} a_n(\tau,t_2)H_n(t_2-\tau, x-\beta(\tau))d\beta(\tau) ,$$

$$n = 0,1,\ldots ; 0 \le t_1 \le t_2$$

then

$$\sum_{n=0}^{N} F_n(t_1,t_2,x)$$

tends to $F(t_1,t_2,x)$ in $L^2(\Omega)$, in the sense that

$$\lim_{N\to\infty}\int_{-\infty}^{\infty} \mathbf{E}\left|F(t_1,t_2,x) - \sum_{n=0}^{N} F_n(t_1,t_2,x)\right|^2 e^{-\frac{x^2}{2(t_2-t_1)}}dx = 0 , 0 \le t_1 \le t_2$$

$$\lim_{N\to\infty}\int_{-\infty}^{\infty} \mathbf{E}\left|\frac{\partial}{\partial x}F(t_1,t_2,x) - \sum_{n=0}^{N}\frac{\partial}{\partial x}F_n(t_1,t_2,x)\right|^2 e^{-\frac{x^2}{2(t_2-t_1)}}dx = 0 , 0 \le t_1 \le t_2 .$$

Since these conditions imply that

$$\sum_{n=0}^{N} F_n(t_1,t_2,\beta(t_2))$$

tends to $F(t_1,t_2,\beta(t_2))$ in $L^2(\Omega)$ for large N, $0 \le t_1 \le t_2$, and since, by Lemma II, for $n = 0,1,\ldots$

$$\int_{t_1}^{t_2} f_n(\tau,t_2)d\beta(\tau) = F_n(t_1,t_2,\beta(t_2)) \ , \ 0 \le t_1 \le t_2 \ ,$$

the proof of Theorem 4.A is complete.

■

As a corollary of Theorem 3.B we present the following result.

Theorem 4.B:

Let

$$\eta(t_1,t_2) = \int_{t_1}^{t_2} \phi(\tau,t_2,\beta(t_2) - \beta(\tau))d\beta(\tau)$$

where

$$\phi(t_1,t_2,x),\ \frac{\partial}{\partial x}\phi(t_1,t_2,x),\ \left(\frac{\partial}{\partial t_1} + \frac{1}{2}\frac{\partial^2}{\partial x^2}\right)\phi(t_1,t_2,x),\ \left(\frac{\partial}{\partial t_2} + \frac{1}{2}\frac{\partial^2}{\partial x^2}\right)\phi(t_1,t_2,x)$$

satisfy (4.1) and (4.2). Then

$$\partial_{t_2}\eta(t_1,t_2) = [\phi(t_2,t_2,0) + \int_{t_1}^{t_2} \frac{\partial}{\partial x}\phi(\tau,t_2,\beta(t_2) - \beta(\tau))d\beta(\tau)]d\beta(t_2)$$

$$+ [\frac{\partial}{\partial x}\phi(t_2,t_2,0) + \int_{t_1}^{t_2} \left(\frac{\partial}{\partial t_2} + \frac{1}{2}\frac{\partial^2}{\partial x^2}\right)\phi(\tau,t_2,\beta(t_2) - \beta(\tau))d\beta(\tau)]dt_2 \ , \ 0 \le t_1 \le t_2 \ .$$

Proof:

The result follows directly from Theorem 3.B once we observe that, by Ito's Formula,

$$\partial_{t_2}\phi(t_1,t_2,\beta(t_2) - \beta(t_1))$$

$$= \frac{\partial}{\partial x}\phi(t_1,t_2,\beta(t_2) - \beta(t_1))d\beta(t_2) + \left(\frac{\partial}{\partial t_2} + \frac{1}{2}\frac{\partial^2}{\partial x^2}\right)\phi(t_1,t_2,\beta(t_2) - \beta(t_1))dt_2 \ .$$

■

§5. ITO-VOLTERRA EQUATION

In this section we study the behavior of the solution to the problem

$$\text{(I-V)} \qquad \xi(t) - \int_0^t \sigma(\tau,t)\xi(\tau)d\beta(\tau) - \int_0^t b(\tau,t)\xi(\tau)d\tau = F(t)$$

where σ, b, F are functions. A more general class of equations is analyzed in Berger [2], [3]; but to make this exposition self-contained, the existence-uniqueness results for (I-V) are presented here.

Theorem 5.A:

Let σ. b, F be functions satisfying

$$\sup_{0 \le t_1 \le t_2 \le T} |\sigma(t_1,t_2)|^2 \equiv \|\sigma\|_T < \infty$$

$$\sup_{0 \le t_1 \le t_2 \le T} |b(t_1,t_2)|^2 \equiv \|b\|_T < \infty$$

$$\sup_{0 \le t \le T} |F(t)|^2 \equiv \|F\|_T < \infty$$

for each $T \ge 0$. Then there exists a solution $\xi(t)$ of (I-V) on $[0,T]$ for any $T \ge 0$ such that

$$\text{(5.1)} \qquad \sup_{0 \le t \le T} \mathbb{E}\, |\xi(t)|^2 < \infty \ .$$

Furthermore, if $\tilde{\xi}(t)$ is another solution of (I-V) satisfying (5.1), then $\tilde{\xi}$ is a version of ξ.

Proof:

To establish existence we construct the successive approximates to (I-V). Thus let

$$\xi_0(t) = F(t) \quad , \quad t \ge 0$$

$$\text{(5.2)} \qquad \xi_n(t) = F(t) + \int_0^t \sigma(\tau,t)\xi_{n-1}(\tau)d\beta(\tau) + \int_0^t b(\tau,t)\xi_{n-1}(\tau)d\tau,$$

$$n = 1,2,\ldots \ ; \ t \ge 0 \ .$$

The first property of these iterates we establish is

(5.3) $$\sup_{0 \le t \le T} \mathbb{E}\,|\xi_n(t)|^2 < \infty \quad , \quad n = 1,2,\dots \; ; \; T \ge 0 \quad .$$

This is shown by induction as follows.

$$\sup_{0 \le t \le T} \mathbb{E}\,|\xi_n(t)|^2 \le 3\|F\|_T^2 + 3N_T \sup_{0 \le t \le T} \mathbb{E}\,|\xi_{n-1}(t)|^2 \; , \; T \ge 0$$

where

$$N_T = \sup_{0 \le t \le T} \Big[\int_0^t |\sigma(\tau,t)|^2 d\tau + t\int_0^t |b(\tau,t)|^2 d\tau\Big] \; , \; T \ge 0 \quad .$$

The next property we establish is

(5.4) $$\sup_{0 \le t \le T} \mathbb{E}\,|\xi_{n+1}(t) - \xi_n(t)|^2 \le 2N_T(1 + \|F\|_T^2)\,\frac{(2M_T)^n}{n!} \; , \; T \ge 0$$

where

$$M_T = \|\sigma\|_T^2 + T\|b\|_T^2 \quad .$$

This is shown by the following observation

$$\mathbb{E}\,|\xi_{n+1}(t) - \xi_n(t)|^2 \le 2M_T \int_0^t \mathbb{E}\,|\xi_n(\tau) - \xi_{n-1}(\tau)|^2 d\tau, \; n = 1,2,\dots \; ; \; 0 \le t \le T \quad .$$

Thus, by (5.4), for each $t \in [0,T]$, the sequence $\xi_n(t)$ converges in $L^2(\Omega)$ to a random variable $\xi(t)$. The process $\xi(t)$ is $\mathfrak{F}(0,t)$-measurable and

$$\sup_{0 \le t \le T} \mathbb{E}\,|\xi(t)|^2 < \infty \quad .$$

Since

$$\lim_{n\to\infty} \sup_{0 \le t \le T} \mathbb{E}\,|\xi_n(t) - \xi(t)|^2 = 0$$

taking limits in (5.2) is valid, and $\xi(t)$ *is, therefore a solution of (I-V).*

To establish uniqueness let $\xi(t)$ and $\tilde{\xi}(t)$ denote two solutions of (I-V) satisfying (5.1). Then

$$\mathbf{E}\,|\xi(t) - \tilde{\xi}(t)|^2 \le 2M_T \int_0^t \mathbf{E}\,|\xi(\tau) - \tilde{\xi}(\tau)|^2 d\tau$$

and thus

$$\mathbf{E}\,|\xi(t) - \tilde{\xi}(t)|^2 = 0 \ , \ t \ge 0 \ .$$

■

The successive approximates (5.2) are particularly interesting in view of the Correction Formula. In fact the solution to (I-V) can be represented as an adapted stochastic integral. This is the content of the following

Theorem 5.B (Resolvent Formula):

Let σ, b, F be as in Theorem 5.A, and also satisfy

(5.5) $$\sup_{0 \le t_1 \le t_2 \le T} \left[\left|\frac{\partial}{\partial t_1} \sigma(t_1,t_2)\right|^2 + \left|\frac{\partial}{\partial t_1} b(t_1,t_2)\right|^2 \right] < \infty \ , \ T \ge 0$$

(5.6) $$\sup_{0 \le t \le T} \left|\frac{d}{dt} F(t)\right|^2 < \infty \ , \ T \ge 0 \ .$$

Define the iterates σ_n, b_n as follows:

$$\sigma_1(t_1,t_2) = \sigma(t_1,t_2) \ , \ b_1(t_1,t_2) = b(t_1,t_2) \ , \ 0 \le t_1 \le t_2$$

$$\sigma_{n+1}(t_1,t_2) = \int_{t_1}^{t_2} \sigma_n(t_1,\tau)\sigma(\tau,t_2)d\beta(\tau) + \int_{t_1}^{t_2} \sigma_n(t_1,\tau)b(\tau,t_2)d\tau \ ,$$

$$b_{n+1}(t_1,t_2) = \int_{t_1}^{t_2} b_n(t_1,\tau)\sigma(\tau,t_2)d\beta(\tau) + \int_{t_1}^{t_2} b_n(t_1,\tau)b(\tau, t_2)d\tau \ ,$$

$$n = 1,2,\ldots \ ; \ 0 \le t_1 \le t_2 \ .$$

Then the resolvents

$$r_\sigma(t_1,t_2) = \sum_{n=1}^{\infty} \sigma_n(t_1,t_2) \ , \ r_b(t_1,t_2) = \sum_{n=1}^{\infty} b_n(t_1,t_2) \ , \ 0 \le t_1 \le t_2$$

exist and are $L_+^{2,1}$-adapted processes. Furthermore the solution to (I-V) is

$$\xi(t) = F(t) + \int_0^t r_\sigma(\tau,t)F(\tau)d\beta(\tau)$$

(5.7)

$$+ \int_0^t [r_b(\tau,t) - \sigma(\tau,\tau)r_\sigma(\tau,t)]F(\tau)d\tau \ , \ t \geq 0 \ .$$

Proof:

This result is actually a corollary of Theorem 5.A. Indeed, by the Correction Formula, it follows that the successive approximates ξ_n are given by

$$\xi_n(t) = F(t) + \int_0^t [\sum_{k=1}^n \sigma_n(\tau,t)]F(\tau)d\beta(\tau)$$

(5.8)

$$+ \int_0^t [\sum_{k=1}^n b_n(\tau,t) - \sigma(\tau,\tau)\sum_{k=1}^{n-1} \sigma_n(\tau,t)]F(\tau)d\tau$$

$$n = 2,3\ldots \ ; \ t \geq 0 \ .$$

Thus the convergence of the approximates implies the existence of r_σ, r_b. The conditions (5.5) and (5.6), together with the *continuity condition of Theorem 2.B, allow us to take limits* in (5.8).

■

Actually, because of the restrictive L^∞ assumptions on σ, b, the convergence of the approximates ξ_n is almost sure convergence. This is because there exists a function $C(t)$ such that

$$\mathbf{E}\,|\xi_{n+1}(t) - \xi_n(t)|^2 \leq \frac{C^n(t)}{n!} \ , \ t \geq 0 \ .$$

This is actually the content of (5.4). And thus the series

$$\sum_{n=1}^{\infty} \mathbf{P}\{|\xi_{n+1}(t) - \xi_n(t)| > \frac{1}{n^2}\}$$

converges for $t \geq 0$. So that by the Borel-Cantelli Lemma, $\xi_n(t)$ converges almost surely for each $t \geq 0$. Similarly the conditions (5.5) and (5.6) imply the almost sure convergence of the terms in (5.8). And thus the Resolvent Formula provides trajectory-type information. For examples concerning the use of the Resolvent Formula, and for additional information about the solution of (I-V), and for the case where σ, b, F are processes themselves, the reader is referred to Berger [2], [3].

REFERENCES

1. Arnold, L., "Stochastic Differential Equations: Theory and Applications." Wiley-Interscience, New York, 1974.
2. Berger, M. A., Stochastic Ito-Volterra Equations. Ph.D. Dissertation, Carnegie-Mellon University, Pittsburgh, Pennsylvania, 1977.
3. Berger, M. A., Ito-Type Integral Operators. To appear.
4. Bharucha-Reid, A. T., "Random Integral Equations." Academic Press, New York, 1972.
5. Friedman, A., "Stochastic Differential Equations and Applications, Vol. 1.", Academic Press, New York, 1975.
6. Gihman, I. I., and Skorohod, A. V., "Stochastic Differential Equations." Springer-Verlag, New York, 1972.
7. Ito, K., Stochastic Integral. Proc. Imperial Acad., Tokyo 20 (1944), 519-524.
8. Ito, K., Multiple Wiener Integral. J. Math. Soc. Japan 3 (1951), 157-169.
9. McKean, H. P., "Stochastic Integrals." Academic Press, New York, 1969.
10. McShane, E. J., "Stochastic Calculus and Stochastic Models." Academic Press, New York, 1974.
11. Meyer, P. S., "Seminaire de Probabilities X, Lecture Notes in Math. 511." Springer-Verlag, New York, 1976.
12. Skorohod, A. V., "Studies in the Theory of Random Processes." Addison-Wesley, Reading, Massachusetts, 1965.
13. Tsokos, C. P., and Padgett, W. J., "Random Integral Equations with Applications to Life Sciences and Engineering." Academic Press, New York, 1974.

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER TSR #1826	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER 14 MRC-TSR-1826
4. TITLE (and Subtitle) 6 A FUBINI THEOREM FOR ITERATED STOCHASTIC INTEGRALS		5. TYPE OF REPORT & PERIOD COVERED Summary Report - ~~no specific~~ reporting period
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) 10 Marc A. Berger		8. CONTRACT OR GRANT NUMBER(s) 15 DAAG29-75-C-0024, NSF-MCS75-17385-A01.
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of Wisconsin 610 Walnut Street Madison, Wisconsin 53706		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Work Unit Number 4 - Probability, Statistics & Combinatorics
11. CONTROLLING OFFICE NAME AND ADDRESS See Item 18.		12. REPORT DATE 11 February 1978
		13. NUMBER OF PAGES 33 12 39p
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office) 9 Technical summary rept.,		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

U.S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

National Science Foundation
Washington, D. C.
20550

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Brownian Motion, Stochastic Integral, Volterra Equation.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This report extends the stochastic integral of Ito to allow for a certain class of anticipating integrands. Probabilistic and computational results concerning this extension are presented. And iterated integrals are discussed.

The motivation for this extension stems from the Ito-Volterra equation. This equation arises from feedback in the presence of white noise, and cannot be inverted using classical stochastic integrals. The inversion involving the extended integrals appears at the end of the report.

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED 221200 B

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)